2

Single-Factor Longitudinal Models for Single-Indicator Data

2.1 INTRODUCTION

In Chapter 1, I introduced the basic concepts of latent state–trait revised (LST-R) theory. These concepts rested on a rather small and unrestrictive set of assumptions (e.g., positive and finite variances of observed variables). LST-R theory allows us to define latent state, trait, state residual, and measurement error variables based on conditional expectations of observed variables and to examine their properties. The definitions of the latent variables that I presented in Chapter 1 are useful because they provide us with latent variables that are relevant to longitudinal studies and make clear what the latent variables mean.

On the other hand, the definitions by themselves do not result in identified and testable statistical models for longitudinal data analysis. This can be easily seen from the fact that according to LST-R theory, each observed variable at each time point is decomposed into its own trait, state residual, and measurement error variable. Our observed data do not contain enough information to identify all of the relevant latent variable means, variances, and covariances. Specific *longitudinal measurement models* introduce simplifying restrictions that reduce the number of latent variables as well as the number of parameters to be estimated.

In this chapter, I introduce very simple longitudinal measurement models with just a single latent factor and a single indicator per measurement occasion. I show how longitudinal measurement models for a single repeatedly measured variable can be obtained by making assumptions, for example, about the homogeneity and (in)dependence of certain latent variables. These assumptions reduce the number of latent variables that need to be considered in a model. They also lead to (1) identified statistical models that can be estimated based on observed data and (2) testable restrictions that allow us to potentially falsify these assumptions based on tests of model fit.

In this chapter as well as in Chapter 3, I focus on single-indicator data, that is, models for longitudinal designs that use only a single repeatedly measured observed variable Y_t for each construct at each time point t. Chapter 3 deals with single-indicator longitudinal models that use more than one latent factor. Many of the models discussed in Chapters 2 and 3 have counterparts for multiple-indicator data, which I discuss in Chapter 5.

2.2 THE RANDOM INTERCEPT MODEL

2.2.1 Introduction

The *random intercept model* is one of the simplest and most restrictive measurement models that can be fit to longitudinal data. It assumes that individuals' trait scores do not change across time. Therefore, it is frequently used as a baseline model in longitudinal analyses. If the random intercept model shows a good fit to the data at hand, this may indicate that a construct did not change across time. In this case, more complex longitudinal models may not be needed.

2.2.2 Model Description

The random intercept model for a single construct and four time points is depicted in Figure 2.1 as a path diagram. In the path diagram, the triangle represents a constant of 1.0 that serves to add the mean structure (means and constant intercepts) to the model. It can be seen that in this model, the repeatedly measured observed variable Y_t loads onto a single common trait factor ξ. (I dropped the index i for the observed variable throughout this chapter, given that all models in this chapter assume only a single repeatedly measured observed variable.) All factor loadings are fixed to 1, and all measurement intercepts are fixed to zero. Hence, the model can be described by the following measurement equation:

$$Y_t = \xi + \varepsilon_t$$

This model is called a *random* intercept model because there is no *fixed* intercept (i.e., additive constant) in the model equation. Instead, there is only a

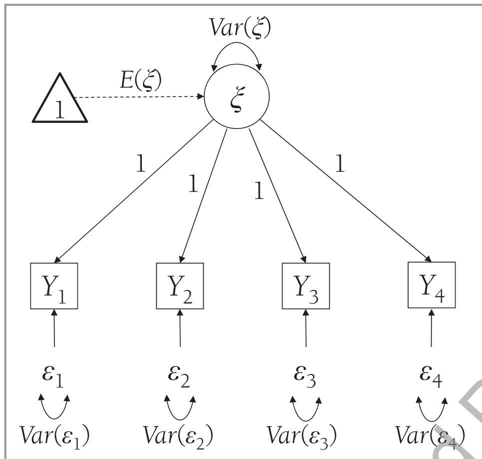

FIGURE 2.1. Path diagram of the random intercept model for a single observed variable Y_t that is measured on four time points. ξ = trait (random intercept) factor; ε_t = measurement error variable.

random intercept that is represented by the trait factor ξ. The trait factor is considered a random intercept because its values can vary across individuals (fixed intercepts, in contrast, are *constants* that do not vary between individuals).

The random intercept model estimates the following parameters:

- the trait factor mean $E(\xi)$,
- the trait factor variance $Var(\xi)$, and
- n measurement error variances $Var(\varepsilon_t)$, one for each for each time point $t = 1, \ldots, n$.

Therefore, the model in general has $n + 2$ free model parameters, where n indicates the total number of measurement occasions. In our example, we have four measurement occasions ($n = 4$). Therefore, our model has $4 + 2 = 6$ free parameters.

In our example with four time points, we have 14 pieces of available information (four Y_t variances, four Y_t means, and six unique Y_t covariances). The random intercept model in our example therefore has $14 - 6 = 8$ degrees of freedom (*df*; for general information on how to calculate the number of pieces of available information and *df* for single-indicator longitudinal models, see Box 2.1).

Readers familiar with classical test theory (CTT) models may find that the random intercept model resembles the model of tau equivalence in CTT, which

BOX 2.1. *Available Information, Model Degrees of Freedom, and Model Identification in Single-Indicator Longitudinal Designs*

What information is used to estimate unknown model parameters in single-indicator longitudinal structural equation models, and how are a model's degrees of freedom (*df*) calculated? In single-indicator longitudinal designs, we only have one (repeatedly observed) measure ($m = 1$) but $n > 1$ time points. We can therefore draw on n repeatedly measured variables $Y_1, \ldots, Y_n$. Each measured variable Y_t has a mean $E(Y_t)$, a variance $Var(Y_t)$, and covariances $Cov(Y_t, Y_s)$, $t \neq s$, with all other measured variables. The observed Y_t means, variances, and covariances provide the information that we use to estimate the unknown model parameters in all longitudinal structural equation models described in this book. In empirical applications, we can compute the means, variances, and covariances for all Y_t variables based on our sample data.

In single-indicator designs, we have n observed Y_t variances, n observed Y_t means, and $0.5 \cdot (n^2 - n)$ observed unique Y_t covariances from which we can estimate unknown model parameters. In total, there are $1.5 \cdot n + 0.5 \cdot n^2$ pieces of available information in single-indicator longitudinal designs. With one measure and four time points, we have $1.5 \cdot 4 + 0.5 \cdot 16 = 14$ pieces of available information (four Y_t variances, four Y_t means, and six unique Y_t covariances).

The degrees of freedom (*df*) of a model are calculated as the number of pieces of available information minus the number of free (unknown) model parameters. Models with negative (< 0) *df* are underidentified models (there is not enough information available to estimate all unknown parameters). For example, the random intercept model is underidentified when there is only a single time point ($n = 1$). Models in which $df = 0$ may be just identified ("saturated"). Saturated models cannot be tested because they always fit perfectly. An example of a saturated model is the random and fixed intercepts model (described in Section 2.3) for $n = 2$ time points. Models with $df > 0$ may be overidentified and contain testable restrictions. Only overidentified models can be empirically falsified (e.g., based on a chi-square test of model fit). For example, the random intercept model is overidentified for $n \geq 2$ time points.

Unfortunately, $df \geq 0$ is only a necessary, but not a sufficient, condition for model identification. That is, models with zero or positive *df* may still be underidentified. More information on the general issue of structural equation model identification can be found, for example, in Bollen (1989).

BOX 2.2. *Defining the Random Intercept Model Based on LST-R Theory*

The random intercept model can be defined using concepts of LST-R theory in a similar way as the tau-equivalence model can be defined using concepts of CTT. Three assumptions are required:

1. *ξ-equivalence:* The latent trait variables for all time points are identical, such that $\xi_t = \xi_s = \xi$ for all $t, s = 1, \ldots, n$.
2. *No situation or person × situation interaction influences:* All latent state residual variables are zero, $\zeta_t = \zeta_s = 0$ for all $t, s = 1, \ldots, n$.
3. *Linear independence of trait and error variables:* $Cov(\xi, \varepsilon_t) = 0$ for all $t = 1, \ldots, n$.

Assumption 1 implies that individuals' trait scores do not change across time. Hence, there is perfect stability of latent trait scores in this model. Figure 2.2 illustrates the model-implied patterns of trait scores for three hypothetical individuals. It can be seen that each individual trajectory is flat, indicating the perfect intraindividual stability of trait scores for all individuals. In actual data, this pattern may be plausible (at least approximately) for rather stable personality constructs such as intelligence.

Assumption 2 implies that measurements reflect only trait (person) aspects of a construct and random measurement error. From the perspective of LST-R theory, the model is restrictive because it assumes that there are no situational influences or person × situation interaction effects that could also have an impact on the measurements. Any intraindividual differences that occur in observed Y_t scores across time are solely due to measurement error according to this model. This implies that the model would only be suitable for perfectly stable trait-like constructs that are unaffected by situational effects (or measures that reflect only the trait aspects of a construct).

Assumption 3 prohibits any correlations between measurement error variables and the common latent trait factor, which is a standard assumption made in most structural equation models (and the default in Mplus). Notice that we do not need to make the assumption of uncorrelated errors, $Cov(\varepsilon_t, \varepsilon_s) = 0$ for $t \neq s$, because this property already follows *by definition* of the error variables in LST-R theory (see Chapter 1, Box 1.2). That is, in all LST-R models, measurement error variables pertaining to different time points are uncorrelated by definition. (In Mplus *all* error variables are uncorrelated by default.)

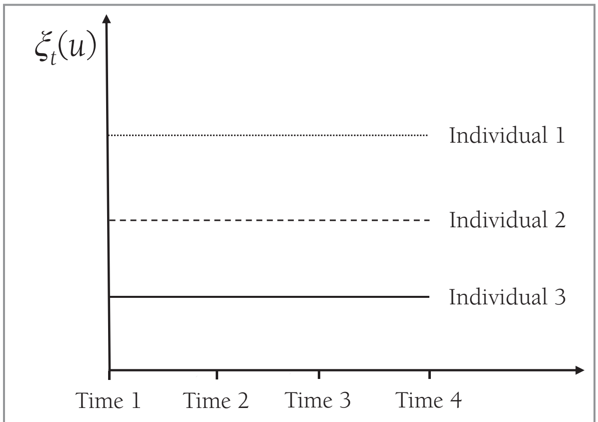

FIGURE 2.2. Illustration of possible model-implied trait scores in the random intercept model for three hypothetical individuals. It can be seen that the model implies perfect stability (no changes) in the trait scores for all individuals.

also represents a single-factor model with equal loadings and equal intercepts. This is not a coincidence as I show in Box 2.2, where I describe how the random intercept model can be derived based on the concepts of LST-R theory. Figure 2.2 shows the model-implied trajectories of the trait scores for three hypothetical individuals. We can see that the model implies perfect stability of individuals' trait scores over time.

Although very restrictive, the random intercept model serves at least three useful purposes in longitudinal studies. First, it can be seen as a baseline "no change" model. If this model fits well, no further analyses of trait changes are needed (assuming sufficient statistical power for tests of model fit). In contrast, if this model is rejected based on tests of model fit, this may indicate that further analyses of change may be useful. Second, the model may be appropriate for social science constructs that show a high degree of stability (i.e., mean and covariance stability) across time and are not prone to situational influences (e.g., intelligence). Third, for such stable constructs, the model allows researchers to estimate the reliability (measurement precision) of the observed scores.

2.2.3 Variance Decomposition and Reliability Coefficient

In the random intercept model, the variance of each measured variable can be decomposed additively because trait and error variables are uncorrelated:

$$Var(Y_t) = Var(\xi) + \mathrm{Var}(\varepsilon_t)$$

A coefficient can then be defined that quantifies the proportion of reliable (systematic) variability in a measured variable (as opposed to unsystematic measurement error variance). This coefficient is known from CTT as the reliability coefficient $Rel(Y_t)$. In the random intercept model, the $Rel(Y_t)$ coefficient is given by

$$\begin{aligned} Rel(Y_t) &= Var(\xi) \,/\, Var(Y_t) \\ &= Var(\xi) \,/\, [Var(\xi) + Var(\varepsilon_t)] \\ &= 1 - \{\mathrm{Var}(\varepsilon_t) \,/\, [Var(\xi) + Var(\varepsilon_t)]\} \end{aligned}$$

The interpretation of the $Rel(Y_t)$ coefficient parallels the interpretation in CTT. Ranging between 0 and 1, values of $Rel(Y_t)$ closer to 1 indicate higher reliability. For example, a value of $Rel(Y_t) = .8$ indicates that 80% of the observed score variability reflect individual differences in true trait scores and the remaining 20% reflect variability due to random measurement error. When the standardized solution (`STDYX`) is requested, Mplus prints the $Rel(Y_t)$ coefficient under `R-SQUARE Observed variable` in the output.

2.2.4 Mplus Application

Below I present an Mplus application of the random intercept model to a hypothetical data set from $N = 300$ individuals who took an intelligence test on four measurement occasions. Individuals' IQ scores at each time point are represented by four observed variables Y_1 through Y_4 in the Mplus analysis. The random intercept model can be specified in Mplus using the following `MODEL` statement (the complete Mplus input and output files along with the example data sets can be found on the companion website [see the box at the end of the table of contents]).

```
MODEL: KSI by Y1-Y4@1;
   [Y1-Y4@0];
   [KSI*];
```

The first line of code specifies that a single factor (latent variable) labeled `KSI` is measured by all four Y_t variables. All four factor loadings are fixed to 1, as indicated by the `@1` statement. The second line of code `[Y1–Y4@0];` sets all four intercepts to zero. (Mplus would by default freely estimate a constant intercept for each measured variable Y_t.) Fixing all loadings to unity and all intercepts to zero is a consequence of Assumption 1 (ξ-equivalence) above. According to the ξ-equivalence assumption, all four time-specific trait factors ξ_t are identical.

Therefore, there are no additive constants (fixed intercepts) in the measurement equation $Y_t = \xi + \varepsilon_t$. In addition, no differences in scaling are assumed between different trait variables ξ_t and $\xi_{t'}$, so that the implicit multiplicative coefficient is 1.0: $\xi_t = 1 \cdot \xi_{t'} = 1 \cdot \xi$. Therefore, all factor loadings must be fixed to 1 in this model. (Mplus by default estimates all factor loadings freely except for the first observed variable listed in the BY statement.)

The mean of the trait factor KSI is estimated by stating the latent variable name in brackets in the third line of code: [KSI*];. The KSI factor variance as well as the four measurement error variances are estimated by default in Mplus and thus do not have to be mentioned explicitly in the MODEL statement.

```
OUTPUT: SAMPSTAT STDYX;
```

As for most analyses in Mplus, it is useful to request that the sample (descriptive) statistics (SAMPSTAT) as well as the completely standardized solution (STDYX) be printed in the output in addition to the default output. Among other information, the standardized solution STDYX provides the reliability estimates $Rel(Y_t)$ as I show below.

Below are selected goodness-of-fit statistics (for a detailed description of these statistics, see Box 2.3) that Mplus computed for the random intercept model:

```
MODEL FIT INFORMATION
. . .
Chi-Square Test of Model Fit
          Value                                5.347
          Degrees of Freedom                       8
          P-Value                             0.7199
RMSEA (Root Mean Square Error Of Approximation)
          Estimate                             0.000
          90 Percent C.I.                      0.000  0.050
          Probability RMSEA >= .05             0.948
CFI/TLI
          CFI                                  1.000
          TLI                                  1.001
. . .
SRMR (Standardized Root Mean Square Residual)
          Value                                0.052
```

The model shows an appropriate fit according to all indices. This is indicated by the large p-value associated with the chi-square test of model fit (p = .7199) as well as RMSEA equal to zero, small SRMR (0.052), and CFI/TLI of 1. An excellent fit is expected here, given that the model was correctly specified for

BOX 2.3. *Model Fit Assessment and Model Comparisons*

One advantage of all models presented in this book is that they can be tested against the observed data and therefore potentially be falsified in empirical research. Model goodness-of-fit assessment in longitudinal confirmatory factor analysis (CFA) and structural equation modeling (SEM) follow general principles that are the same as those for other types of CFA and SEM approaches. The basic idea of model fit assessment in CFA or SEM is that a model can only be correct if the covariance and mean structure in the population match the model-implied covariance and mean structure. A chi-square test of model fit is frequently used to test the null hypothesis that the data structure in the population is identical to the model-implied data structure. This null hypothesis of exact model fit can be rejected when the chi-square test returns a small p-value (e.g., $p \leq .05$). In other words, a small p-value can lead to model rejection.

The null hypothesis tested with the chi-square test is rather strict, requiring an exact fit of a given model in the population. Some researchers (e.g., Bollen, 1989) therefore view the null hypothesis as unrealistic for most social science applications. This is because models by definition represent simplifications of reality and are thus not expected to show a perfect fit. To examine and quantify "approximate" model fit, a number of fit indices have been developed that are also frequently reported in the literature. The most commonly used indices include the root mean square error of approximation (RMSEA; Steiger & Lind, 1980), comparative fit index (CFI; Bentler, 1990), and standardized root mean square residual (SRMR; Bentler, 1995; Jöreskog & Sörbom, 1981). The simultaneous consideration of multiple indices has been recommended, where RMSEA values close to .06, CFI values close to .95, and SRMR values close to .08 may indicate adequate approximate model fit (Hu & Bentler, 1999). In addition, for a given data set, different models can be compared based on information criteria (IC) such as the Bayesian information criterion (BIC). In such comparisons, models with lower BIC values are preferred. A detailed discussion of model goodness-of-fit indices for confirmatory factor and structural equation models can be found in Bollen and Long (1993), Hu and Bentler (1999), as well as Schermelleh-Engel, Moosbrugger, and Müller (2003).

simulated data drawn from a known population model. Next, Mplus provides the maximum likelihood estimated unstandardized and standardized parameter estimates for the random intercept model:

```
MODEL RESULTS
                                                    Two-Tailed
                    Estimate       S.E.  Est./S.E.    P-Value
 KSI      BY
    Y1                 1.000      0.000    999.000    999.000
    Y2                 1.000      0.000    999.000    999.000
    Y3                 1.000      0.000    999.000    999.000
    Y4                 1.000      0.000    999.000    999.000
 Means
    KSI              100.279      0.850    118.027      0.000
 Intercepts
    Y1                 0.000      0.000    999.000    999.000
    Y2                 0.000      0.000    999.000    999.000
    Y3                 0.000      0.000    999.000    999.000
    Y4                 0.000      0.000    999.000    999.000
 Variances
    KSI              210.102     17.689     11.878      0.000
 Residual Variances
    Y1                26.103      2.847      9.168      0.000
    Y2                30.085      3.141      9.579      0.000
    Y3                25.683      2.812      9.133      0.000
    Y4                22.365      2.578      8.675      0.000

STANDARDIZED MODEL RESULTS (STDYX Standardization)
                                                    Two-Tailed
                    Estimate       S.E.  Est./S.E.    P-Value
 KSI      BY
    Y1                 0.943      0.007    131.911      0.000
    Y2                 0.935      0.008    119.133      0.000
    Y3                 0.944      0.007    131.637      0.000
    Y4                 0.951      0.007    144.281      0.000
 Means
    KSI                6.918      0.297     23.289      0.000
 Intercepts
    Y1                 0.000      0.000    999.000    999.000
    Y2                 0.000      0.000    999.000    999.000
    Y3                 0.000      0.000    999.000    999.000
    Y4                 0.000      0.000    999.000    999.000
 Variances
    KSI                1.000      0.000    999.000    999.000
 Residual Variances
    Y1                 0.111      0.013      8.194      0.000
    Y2                 0.125      0.015      8.530      0.000
    Y3                 0.109      0.014      8.046      0.000
    Y4                 0.096      0.013      7.679      0.000
```

```
R-SQUARE
    Observed                                                Two-Tailed
    Variable        Estimate       S.E.   Est./S.E.     P-Value
    Y1                 0.889      0.013      65.956       0.000
    Y2                 0.875      0.015      59.567       0.000
    Y3                 0.891      0.014      65.819       0.000
    Y4                 0.904      0.013      72.141       0.000
```

The unstandardized solution provides the estimates of the latent trait factor mean (100.279) and variance (210.102), as well as the estimates of the four freely estimated measurement error variances (printed under `RESIDUAL VARIANCES`, values ranging between 22.365 and 30.085). In this example, the measurement error variances are fairly small compared to the trait factor variance, indicating high reliability (measurement precision). The degree of reliability of the measures becomes clearer from the estimates provided in the standardized solution.

In the standardized solution, we obtain estimates of the standardized factor loadings (range: .935 through .951). Notice that even though we fixed all factor loadings to unity, the standardized factor loadings differ from one (and from each other). This is because we allowed the measurement error variances to differ across time. As a consequence, the observed variable variances are not restricted to be equal by the model. If we had constrained the measurement error variances to be time-invariant, we would have also obtained equal standardized loadings in this model.

The standardized loadings in the example output indicate that the measures Y_t are strongly correlated with the trait factor ξ (all loadings are > .93). Again, this indicates the high reliability of the measurements. The reliability estimates $Rel(Y_t)$ themselves are given by the squared standardized loadings in this model and can be found under `R-SQUARE` in the Mplus output [range: $.875 \leq Rel(Y_t) \leq .904$]. The trait factor ξ thus accounts for ≥ 87.5% of the variability in the observed variables Y_t in this example. This shows that there is not much room for additional situational or person × situation interaction effects, implying that the assumption of a strongly trait-like construct is plausible. (If there is any systematic situational variance in this example, it would be a very small amount and would be confounded with measurement error variance.)

2.2.5 Summary

The random intercept model implies both strict covariance and strict mean stability at the latent level. Any changes in measured scores Y_t are solely attributed to measurement error in this model. Situation-specific fluctuations in the true scores are not allowed. In the sense of LST-R theory, the model implies that the construct under study is perfectly trait-like and stable over time.

The random intercept model is a useful baseline model for longitudinal data, because it allows testing the simple hypothesis that there have not been any true changes in a construct across time. Given sufficient statistical power, a well-fitting random intercept model may indicate that further analyses of change are not needed and that the construct is a stable trait for the time period and population under study. In this case, no further models may need to be examined. The researcher can use the random intercept model to estimate the reliabilities of the indicators and, if desired, link the trait factor to other external variables to study relationships with other constructs.

The random intercept model is overidentified and implies a testable restriction already when there are only two time points. In this case, the model allows testing whether the two observed means are equal in the population, that is, whether $E(Y_1) = E(Y_2)$. For three or more time points, the random intercept model allows testing whether all variables have equal covariances (in addition to having equal means).

In many practical social science applications, the random intercept model does *not* fit as well as in our example, and/or the standardized factor loadings are a lot smaller than the ones obtained here, resulting in seemingly low reliabilities of the observed variables. If the model does not fit, this may be an indication that trait changes did occur (which the model does not allow). If the standardized loadings and reliabilities are weaker than one would expect (e.g., based on known or typical reliabilities for the given measure), this might indicate that the measurement error variances are overestimated due to the presence of systematic situation or person × situation interaction effects (which the model also ignores). If systematic situation or person × situation interaction effects are present in a measure, the random intercept model confounds these effects with measurement error, leading to an inflation of the measurement error variance estimates and an underestimation of the indicator reliabilities (R^2 values).

Despite this misspecification, the random intercept model may show a decent fit to the data. Hence, model fit alone does not tell the researcher whether the random intercept model is reasonable in a given application. The standardized loadings and reliabilities as well as other parameter estimates should also be considered.

The next two models address the first limitation of the random intercept model (no trait changes permitted) to some degree. The trait–state–error model discussed in Section 3.4 as well as the multiple-indicator LST models described in Chapter 5 address the second limitation (no situation and interaction effects permitted).

2.3 THE RANDOM AND FIXED INTERCEPTS MODEL

2.3.1 Introduction

The random intercept model implies that there are no mean changes across time. This is because the model estimates only one common mean parameter $E(\xi)$. The fixed intercepts (additive constants) of the observed variables Y_t are all set to zero, and their loadings are set to 1. Therefore, the model implies mean stability across time. This assumption can be relaxed by adding fixed intercept parameters to the random intercept model. I call the resulting model the *random and fixed intercepts model*, as it contains both a random intercept (trait) factor and fixed intercepts (additive constants α_t).

2.3.2 Model Description

The random and fixed intercepts model is depicted in Figure 2.3. Notice that the random intercept factor ξ_1 now has an index for Time $t = 1$. This is because

BOX 2.4. *Means of Linear Combinations*

The mean structure plays an important role in most longitudinal structural equation models, as researchers are often interested in mean differences between measurement occasions. In the general so-called *congeneric* measurement model that is often used in longitudinal confirmatory factor models, an observed variable Y is connected to a latent variable, say η by an additive constant (intercept α) and a multiplicative constant (factor loading λ):

$$Y = \alpha + \lambda \cdot \eta + \varepsilon$$

where ε indicates a measurement error (residual) variable. The mean (or expectation) $E(.)$ of Y in the congeneric measurement model is given by

$$E(Y) = \alpha + \lambda \cdot E(\eta)$$

because the mean of a constant (α) is equal to the constant itself and ε as a measurement error variable has a mean of zero. Therefore, when $\lambda = 1$ (as in, e.g., the tau-equivalent and tau-parallel measurement models of classical test theory), the mean of Y is equal to the mean of η plus a constant α. When $E(\eta) = 0$ (i.e., when the latent variable is mean centered), the mean of Y is equal to α. The latter case is the default in Mplus 8 for conventional confirmatory factor and structural equation models (latent variable means are set to zero, and observed variable intercepts are estimated as the default).

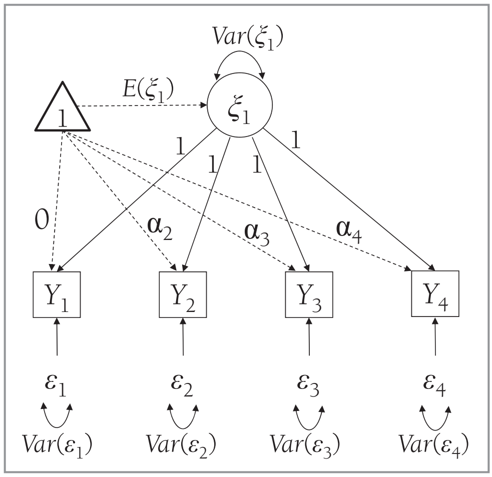

FIGURE 2.3. Path diagram of the random and fixed intercepts model for a single observed variable Y_t that is measured on four time points. ξ_1 = trait factor at Time 1; ε_t = measurement error variable; α_t = constant (fixed) intercept coefficient.

I (arbitrarily) chose Time 1 as the reference time point to make comparisons to. The constant of 1 in the triangle in Figure 2.3 adds both a mean to the reference trait factor ξ_1 and constant intercepts α_i to all Y_t variables except Y_1. For Y_1, the intercept α_1 remains fixed at zero. This makes Y_1 a so-called *reference indicator* and allows us to identify the trait factor mean $E(\xi_1)$ as the mean at Time 1 (the reference time point). This can be seen by applying algebraic rules for means of linear combinations (see Box 2.4). The means of the remaining observed variables Y_2, Y_3, and Y_4 can now be different from the mean of Y_1 because Y_2, Y_3, and Y_4 have additional additive constants α_t. These constants reflect the mean differences relative to Y_1. The model can be described by the following measurement equation:

$$Y_t = \alpha_t + \xi_1 + \varepsilon_t, \text{ where } \alpha_1 = 0$$

The choice of the reference variable for which the intercept remains fixed at zero is arbitrary from a mathematical point of view (different choices of the reference variable lead to the same model-implied covariance and mean structure and result in the same model fit). However, some choices may lead to more readily interpretable results than others. The choice should be based on which mean comparisons are most interesting to the researcher. Often, it will be useful to make comparisons relative to Time 1, which is what I show in the example presented later in this section.

In summary, the random and fixed intercepts model estimates the following parameters:

- the trait factor mean $E(\xi_1)$,
- the trait factor variance $Var(\xi_1)$,
- $n - 1$ intercept constants α_t, $t = 1, \ldots, n$, and
- n measurement error variances $Var(\varepsilon_t)$, one for each for each time point $t = 1, \ldots, n$.

Therefore, the model in general has $2 \cdot n + 1$ free model parameters. In our example with four time points, we thus have $2 \cdot 4 + 1 = 9$ free parameters to estimate. Given that we have 14 pieces of available information (four Y_t variances, four Y_t means, and six unique Y_t covariances), the model has $14 - 9 = 5$ *df*. In Box 2.5, I explain how the random and fixed intercepts model can be defined based on the concepts of LST-R theory.

Figure 2.4 illustrates the types of trait-change patterns that the random and fixed intercepts model permits. It can be seen that the model allows for true changes in the trait scores across time, but the amount of change has to be the same for all individuals. Such strictly homogeneous change would be expected, for example, if an intervention or other event had the exact same effect on all

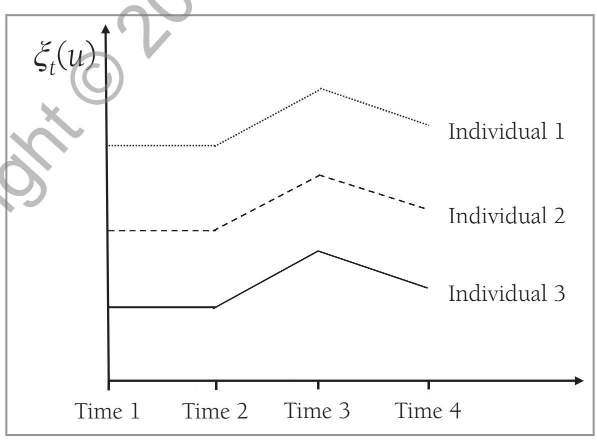

FIGURE 2.4. Illustration of possible model-implied trait-change patterns in the random and fixed intercepts model for three hypothetical individuals. It can be seen that the model implies equal amounts of changes in the trait scores for all individuals. In this example, there is no change between Time 1 and Time 2 (i.e., $\alpha_2 = 0$).

BOX 2.5. Defining the Random and Fixed Intercepts Model Based on LST-R Theory

From the perspective of LST-R theory, the random and fixed intercepts model can be defined by making a slight modification to Assumption 1 made for the random intercept only model presented in Box 2.2:

1. *Essential ξ-equivalence:* Without loss of generality, let ξ_1 be the reference trait variable. The latent trait variables ξ_t for all other time points $t > 1$ differ from ξ_1 only by an additive constant, such that $\xi_t = \alpha_t + \xi_1$ for all $t = 1, \ldots, n$, where α_t denotes a real constant and $\alpha_1 = 0$.
2. *No situation or person × situation interaction influences:* All latent state residual variables are zero, $\zeta_t = \zeta_s = 0$ for all $t, s = 1, \ldots, n$.

Instead of assuming *strict* equivalence of the traits for different time points, we now only assume *essential* equivalence. That is, the trait variables are now allowed to differ but only by an additive constant. This implies that, although the trait (and observed variable) means can differ between time points, individuals' trait scores are still perfectly correlated across time. The second assumption is analogous to the assumption made in the random intercept model (see Box 2.2).

Notice that here we do not need to make an assumption regarding the uncorrelatedness of the trait factor ξ_1 and the error variables ε_t. This is because we chose the trait at Time 1 to serve as a "common" trait factor. According to LST-R theory, a trait variable is by definition uncorrelated with all error variables that are measured at either the same or future time points (see Chapter 1, Box 1.2). Hence, ξ_1 is by definition uncorrelated with all ε_t in this model. However, an assumption about uncorrelated trait and error variables would have to be made if a trait variable other than ξ_1 were chosen as reference to prohibit correlations of the trait variable with prior error variables.

Note that the random and fixed intercepts model is equivalent to the model of essential tau equivalence in CTT that is frequently used in cross-sectional psychometric analyses. The essential tau-equivalence model of CTT assumes that a set of observed variables measure the same construct, but that they can differ in their difficulty (means).

participants or if developmental paths were strictly homogeneous across individuals. Like the random intercept model, the random and fixed intercepts model implies that no situation or person × situation interaction effects are relevant to the given construct or reflected in the measure.

2.3.3 Variance Decomposition and Reliability Coefficient

The variance decomposition and reliability coefficient in the random and fixed intercepts model are analogous to the random intercept model:

$$
\begin{aligned}
Var(Y_t) &= Var(\xi_1) + \mathrm{Var}(\varepsilon_t) \\
Rel(Y_t) &= Var(\xi_1) \,/\, Var(Y_t) \\
&= Var(\xi_1) \,/\, [Var(\xi_1) + Var(\varepsilon_t)] \\
&= 1 - \{\mathrm{Var}(\varepsilon_t) \,/\, [Var(\xi_1) + Var(\varepsilon_t)]\}
\end{aligned}
$$

The reliability coefficient has the same meaning and interpretation as in CTT as well as in the random intercept model.

2.3.4 Mplus Application

Below I discuss an illustrative application of the random and fixed intercepts model in Mplus. I again refer to a fictitious data set from $N = 300$ individuals whose IQ scores were recorded on four measurement occasions and are represented by observed variables Y_1 through Y_4. The random and fixed intercepts model can be specified in Mplus using the following commands:

```
MODEL: KSI1 by Y1-Y4@1;
  [Y1@0 Y2-Y4*];
  [KSI1*];
```

It can be seen that the only difference in the model specification relative to the random-intercept-only model is in the second line of code, which now only sets the first intercept (of Y_1) to zero and estimates the remaining three intercepts (of Y_2, Y_3, and Y_4).

I focus on the unstandardized parameter output for the random and fixed intercepts model because the main new feature of this model lies in the possibility of studying mean differences between time points through estimation of the constant intercept parameters α_t. (The standardized loadings and reliability estimates that are provided as part of the STDYX output are interpreted in the same way as for the random intercept model described in Section 2.2.) The unstandardized output for the random and fixed intercepts model provides the estimates of the latent trait factor mean and variance, observed variable constant intercepts α_t, and estimates of the measurement error variances (RESIDUAL VARIANCES).

```
MODEL RESULTS
                                                          Two-Tailed
                         Estimate       S.E.  Est./S.E.     P-Value
 KSI1      BY
    Y1                      1.000      0.000    999.000     999.000
    Y2                      1.000      0.000    999.000     999.000
    Y3                      1.000      0.000    999.000     999.000
    Y4                      1.000      0.000    999.000     999.000
 Means
    KSI1                  100.107      0.887    112.830       0.000
 Intercepts
    Y1                      0.000      0.000    999.000     999.000
    Y2                     -0.669      0.432     -1.546       0.122
    Y3                     -2.904      0.415     -6.994       0.000
    Y4                     -4.735      0.402    -11.787       0.000
 Variances
    KSI1                  210.102     17.688     11.878       0.000
 Residual Variances
    Y1                     26.056      2.843      9.164       0.000
    Y2                     30.050      3.137      9.578       0.000
    Y3                     25.682      2.812      9.135       0.000
    Y4                     22.357      2.577      8.676       0.000
```


The mean of the trait factor (100.107) in this model represents the trait mean at Time 1. The intercepts α_t for Y_2, Y_3, and Y_4 indicate the trait mean differences relative to the Time-1 trait mean. The intercept for Y_2 is estimated to be –0.669, indicating a smaller mean at Time 2 relative to Time 1. However, this mean difference is not statistically significant at the .05 level as indicated by the standard error and z test for this parameter, $SE = 0.432$, $z = -1.546$, $p = .122$. In other words, the null hypothesis of equal means between Times 1 and 2 cannot be rejected for alpha = .05. In contrast, the estimated intercept for Y_3 (–2.904) is negative and statistically significantly different from zero ($SE = 0.415$, $z = -6.994$, $p < .001$). This indicates that there was a decline in IQ means between Time 1 and Time 3. The Time-4 intercept (–4.735) is also statistically significant ($SE = 0.402$, $z = -11.787$, $p < .001$), indicating that the same trend continues (there is a further decline in average IQ scores beyond Time 3). Overall, from Time 1 to Time 4, each individual showed a “loss” of close to 5 IQ points in this hypothetical study.[1]

2.3.5 Summary

The random and fixed intercepts model can be seen as an extension of the random intercept model. By allowing for an additive constant α_t, the random and fixed intercepts model relaxes the assumption of no mean changes across time made in the random intercept model. It still assumes that there are no *individual differences* in trait change over time and no situation or person × situation interaction effects.

The random and fixed intercepts model is just identified when there are only two time points. In this case, the model is saturated and does not contain testable restrictions. For three or more time points, the model becomes overidentified and allows testing whether the covariances between all Y_t variables are equal in the population.

The random and fixed intercepts model implies that the same true trait score differences that are reflected in the constant intercept parameters α_t apply to *every* individual. This does not seem very realistic for most longitudinal studies (including a population-based study on intelligence in which one would expect that some individuals' IQ scores change more than others' and/or that some individuals show no changes at all). Thus, the random and fixed intercepts model is appropriate only when a researcher hypothesizes that all individuals' trait scores changed by the same amount. The next model addresses the issue of interindividual differences in intraindividual changes to some extent.

2.4 THE ξ-CONGENERIC MODEL

2.4.1 Introduction

Figure 2.4 illustrated the rather restrictive change process implied by the random and fixed intercepts model. According to the model, all individuals changed by the exact same amount. Any additional changes in the observed scores would represent measurement error. This is fairly unrealistic in practice. Individuals often differ in how much they change (i.e., there are true interindividual differences in intraindividual change). The random and fixed intercepts model does not permit a change process in which some individuals' true scores change more than those of other individuals. In the single-factor approach, this limitation can be addressed to some extent by relaxing another implicit constraint made in the two previous models. In this section, I describe a single-factor model that I refer to as the ξ-congeneric model. The ξ-congeneric model freely estimates not only constant intercepts but also factor loadings.

2.4.2 Model Description

The ξ-congeneric model is depicted in Figure 2.5. It can be seen that, in addition to estimating additive constants (intercepts) α_t, the model also permits Y_2, Y_3, and Y_4 to have freely estimated multiplicative constants (factor loadings) λ_t. Y_1 again serves as a reference indicator with $\alpha_1 = 0$ and $\lambda_1 = 1$. Therefore, the scale and mean of the trait factor ξ_1 are identified via the reference indicator Y_1. As a consequence of estimating both intercepts and loadings for Y_2, Y_3, and Y_4, individuals' trait scores at Times 2, 3, and 4 can differ from their ξ_1 score by both an additive constant (α_t) and a multiplicative constant (λ_t):

$$\xi_t = \alpha_t + \lambda_t \cdot \xi_1$$

It is again arbitrary from a mathematical point of view which time point we select to serve as reference point. Here, for consistency, we again select Time 1, making Y_1 the reference indicator. The model is described by the following measurement equation:

$$Y_t = \alpha_t + \lambda_t \cdot \xi_1 + \varepsilon_t, \text{ where } \alpha_1 = 0 \text{ and } \lambda_1 = 1$$

The ξ-congeneric model parallels the model of tau-congeneric variables in CTT (Jöreskog, 1971a). The tau-congeneric model of CTT assumes that a

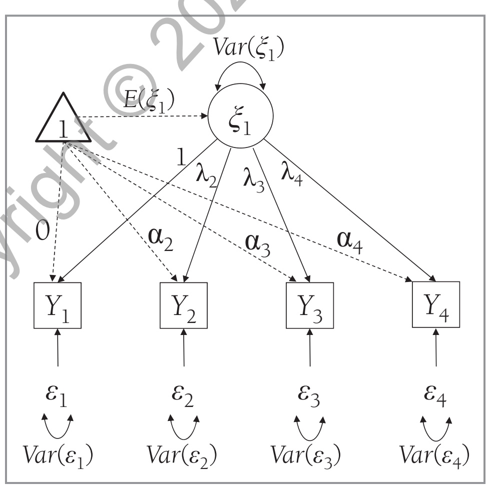

FIGURE 2.5. Path diagram of the ξ-congeneric model for a single observed variable Y_t that is measured on four time points. ξ_1 = trait factor at Time 1; ε_t = measurement error variable; α_t = constant (fixed) intercept coefficient; λ_t = constant factor loading (slope) coefficient.

set of observed variables measure the same construct, but that they can differ in their difficulty (means) as well as in their discrimination (slope) and/or in scaling (the units of measurement can differ between variables). Given its close relationship to the tau-congeneric CTT model, I refer to the model as the *ξ-congeneric model*.

In summary, the ξ-congeneric model estimates the following parameters:

- the trait factor mean $E(\xi_1)$,
- the trait factor variance $Var(\xi_1)$,
- $n - 1$ intercept constants α_t, $t = 1, \ldots, n$,
- $n - 1$ multiplicative constants λ_t, $t = 1, \ldots, n$, and
- n measurement error variances $Var(\varepsilon_t)$, one for each for each time point $t = 1, \ldots, n$.

Therefore, the model in general has $3n$ free model parameters. In our example with four time points, we thus have $3 \cdot 4 = 12$ free parameters to estimate. Given that we have 14 pieces of available information (four Y_t variances, four Y_t means, and six unique Y_t covariances), the model has $14 - 12 = 2$ *df*. In Box 2.6, I explain how the ξ-congeneric model can be defined based on the concepts of LST-R theory.

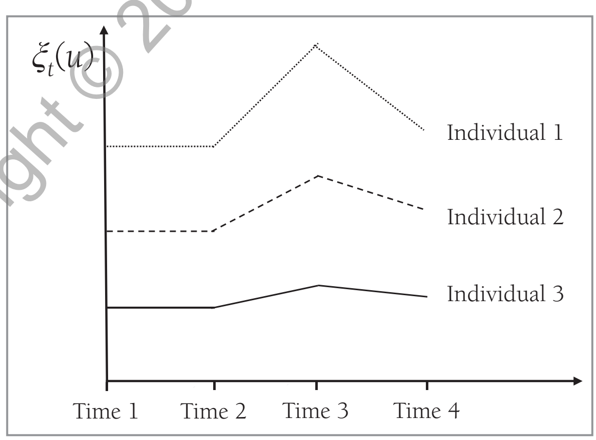

FIGURE 2.6. Illustration of possible model-implied trait-change patterns in the ξ-congeneric model for three hypothetical individuals. It can be seen that, although the model allows for individual differences in the amount of trait change, individuals' trait scores remain perfectly correlated across time. In this example, there is no change between Time 1 and Time 2 (i.e., $\alpha_2 = 0$ and $\lambda_2 = 1$).

Figure 2.6 illustrates the types of change patterns that the ξ-congeneric model permits. Not surprisingly, the ξ-congeneric model also allows for true changes (i.e., changes in the trait scores) across time. In addition, the change patterns are less restrictive than in the random and fixed intercepts model because of the freely estimated factor loadings. Now individuals are allowed to differ from one another also in *how much* their trait scores change. However, there is still an important restriction in the ξ-congeneric model with regard to interindividual differences in change across time. That is, the trait scores remain perfectly correlated across time so that the rank order of individuals remains constant over time.

BOX 2.6. *Defining the ξ-Congeneric Model Based on LST Theory*

Using LST-R theory, we can define the ξ-congeneric model by assuming that the trait variables for different time points are positive linear functions of one another.* This can be referred to as the assumption of *ξ-congenericity:*

1. *ξ-congenericity:* Without loss of generality, let ξ_1 be the reference trait variable. The latent trait variables ξ_t for all other time points $t > 1$ are positive linear functions of the reference trait variable ξ_1, such that $\xi_t = \alpha_t + \lambda_t \cdot \xi_1$ for all $t = 1, \ldots, n$, where α_t and λ_t denote real constants, $\alpha_1 = 0$, $\lambda_1 = 1$, and $\lambda_t > 0$.
2. *No situation or person × situation interaction influences:* All latent state residual variables are zero, $\zeta_t = \zeta_s = 0$ for all $t, s = 1, \ldots, n$.

The trait variables are now allowed to differ from ξ_1 not only by an additive constant (α_t) as under the assumption of essential ξ-equivalence, but also by a multiplicative constant (λ_t). Nonetheless, individuals' trait scores are still perfectly correlated across time even under the weaker assumption of ξ-congenericity, because the trait variables at different time points are still perfectly linearly related according to Assumption 1. The second assumption is the same as in the two previously discussed models.

Again, we do not need to make an assumption about uncorrelated ξ_1 and ε_t, because ξ_1 is by definition uncorrelated with all ε_t according to LST-R theory. (Such an assumption would have to be made if a trait variable other than ξ_1 were selected as a reference trait.)

*It appears unlikely that the trait variables for the same measure would ever be *negatively* correlated across time. Therefore, I assume *positive* linear functions, implying that the trait variables are perfectly positively correlated.

2.4.3 Variance Decomposition and Reliability Coefficient

The variance decomposition in the ξ-congeneric model is given by

$$Var(Y_t) = \lambda_t^2 \cdot Var(\xi_1) + Var(\varepsilon_t)$$

Notice that the multiplicative constant (factor loading) λ_t now has to be taken into account. The ξ-congeneric model also allows estimating indicator reliability at each time point:

$$\begin{aligned} Rel(Y_t) &= \lambda_t^2 \cdot Var(\xi_1) \,/\, Var(Y_t) \\ &= \lambda_t^2 \cdot Var(\xi_1) \,/\, [\lambda_t^2 \cdot Var(\xi_1) + Var(\varepsilon_t)] \\ &= 1 - \{\mathrm{Var}(\varepsilon_t) \,/\, [\lambda_t^2 \cdot Var(\xi_1) + Var(\varepsilon_t)]\} \end{aligned}$$

The reliability coefficient $Rel(Y_t)$ again has the same meaning and interpretation as in previously discussed models.

2.4.4 Mplus Application

Below I describe a hypothetical application of the ξ-congeneric model to N = 300 individuals who provided IQ data on four measurement occasions. The model can be specified in Mplus by using the following commands:

```
MODEL: KSI1 by Y1@1 Y2-Y4*;
  [Y1@0 Y2-Y4*];
  [KSI1*];
```

Notice that the loadings for Y_2, Y_3, and Y_4 in the BY statement are no longer fixed to unity. The output for the ξ-congeneric model provides the estimates of the factor loadings for Y_2, Y_3, and Y_4, latent trait factor mean and variance, observed variable fixed intercepts, and estimates of the measurement error variances:

```
MODEL RESULTS
                                                         Two-Tailed
                    Estimate       S.E.  Est./S.E.    P-Value
 KSI1      BY
    Y1                 1.000      0.000    999.000    999.000
    Y2                 1.517      0.041     37.389      0.000
    Y3                 1.965      0.049     40.214      0.000
    Y4                 2.679      0.063     42.245      0.000
```

```
 Means
    KSI1              100.107       0.870    115.034       0.000
 Intercepts
    Y1                  0.000       0.000    999.000     999.000
    Y2                -46.318       4.102    -11.290       0.000
    Y3                -91.251       4.939    -18.474       0.000
    Y4               -162.264       6.410    -25.316       0.000
 Variances
    KSI1              200.144      18.450     10.848       0.000
 Residual Variances
    Y1                 27.053       2.453     11.027       0.000
    Y2                 34.831       3.516      9.905       0.000
    Y3                 35.856       4.317      8.305       0.000
    Y4                 42.155       6.708      6.284       0.000
```

The mean of the trait factor again represents the mean at Time 1 (100.107). The intercepts for Y_2, Y_3, and Y_4 were all estimated to be negative. The intercepts in this model no longer directly indicate the mean difference relative to the Time-1 mean because the factor loadings (multiplicative constants) now also have to be taken into account when calculating the means from the model parameters—unless all loadings are estimated to be exactly 1.0 (see Box 2.4). The estimated loadings for Times 2, 3, and 4 in this example are all larger than 1.0, so that the mean differences cannot be directly inferred from the intercepts. However, using the formulas presented in Box 2.4, we can compute the means for Times 2, 3, and 4 in Mplus using the `MODEL CONSTRAINT` option (see Box 2.7 on pages 40–41 for details). For this purpose, we change the `MODEL` statement for the ξ-congeneric model in the Mplus input file as follows:

```
MODEL: KSI1 by Y1@1
               Y2* (l2)
               Y3* (l3)
               Y4* (l4);
       [Y1@0];
       [Y2-Y4*] (a2-a4);
       [KSI1*] (E1);
MODEL CONSTRAINT:
NEW(E2 E3 E4);
E2 = a2 + l2*E1;
E3 = a3 + l3*E1;
E4 = a4 + l4*E1;
MODEL TEST:
E1 = E2;
E2 = E3;
E3 = E4;
```

BOX 2.7. *The* MODEL CONSTRAINT *and* MODEL TEST *Options in Mplus*

The Mplus MODEL CONSTRAINT statement is a special feature of the MODEL command that is useful when one wants to constrain parameters in complex ways or compute model parameters that are not directly estimated by Mplus, but that are functions of other model parameters that *are* directly estimated. An example is the ξ-congeneric model, in which a mean is estimated for the first trait variable (ξ_1) but not for the remaining trait variables (ξ_1, ξ_2, and ξ_3). The remaining trait means at Times 2, 3, and 4 are functions of the ξ_1 mean as well as the relevant factor loadings λ_t and intercepts α_t (see Box 2.4 and text). When using MODEL CONSTRAINT to define new parameters, each relevant parameter in the conventional MODEL statement has to be given a label or a number by using parentheses. For example:

```
MODEL:
KSI1 by Y1@1
        Y2 (l2);
[Y2] (a2);
[KSI1] (E1);
```

In this example, l2 is the label chosen for the factor loading λ_2 pertaining to Y_2; a2 is the label chosen for the Y_2 constant intercept α_2; and E1 is the label for the ξ_1 trait factor mean $E(\xi_1)$. These parameters can now be used in MODEL CONSTRAINT to define "new" parameters or to implement complex constraints on existing or new parameters. For example, we can compute the mean of ξ_2 as a new parameter:

```
MODEL CONSTRAINT:
NEW(E2);
E2 = a2 + l2*E1;
```

Here, E2 is the new parameter to be defined that refers to the mean of ξ_2. According to the rule discussed in Box 2.4, E2 is equal to the intercept α_2 (labeled a2) plus the loading λ_2 (labeled l2) times the mean of ξ_1 (labeled E1). Notice that the * symbol refers to multiplication within the MODEL CONSTRAINT option in Mplus.

The MODEL TEST option can be used within the MODEL command to test specific constraints on parameters using a Wald test statistic. For example, it can be tested whether two means are equal:

```
MODEL TEST:
E1 = E2;
```

Mplus then provides a Wald test statistic for the null hypothesis that the two means are equal in the population as part of the `MODEL FIT INFORMATION` section in the output file:

```
Wald Test of Parameter Constraints
          Value                                   78.023
          Degrees of Freedom                           1
          P-Value                                 0.0000
```

In this example, the Wald test statistic is significant ($p < .001$), indicating that the null hypothesis of equality of means can be rejected for alpha = .05.

Notice that in the modified input, I labeled all estimated loadings (`l2`, `l3`, `l4`), intercepts (`a2`, `a3`, `a4`), and the Time-1 trait factor mean (`E1`) by putting the chosen labels in parentheses `()` behind the relevant parameter. Using these labels, I implemented the equations presented in Box 2.4 for the calculation of the Time 2, 3, and 4 trait means in the `MODEL CONSTRAINT` option. To do so, I first listed these means in the `NEW` statement (labeled `E2`, `E3`, and `E4`) to indicate that they are to be defined as new parameters. Subsequently, I defined each parameter. For example, the Time-2 mean was defined as `E2 = a2 + l2*E1`, where the `*` symbol refers to multiplication.

Finally, the `MODEL TEST` option allows us to compute a Wald test of equality of means (see Box 2.7). This test can be conducted for just one pair of means or for a larger number of mean comparisons. In this case, I requested an omnibus Wald test of the null hypothesis that all four means are equal in the population. As a consequence of adding the new parameters through `MODEL CONSTRAINT`, we obtain the means as additional parameters in the unstandardized parameter estimate list (`MODEL RESULTS` section):

```
MODEL RESULTS
. . .
New/Additional Parameters
    E2              105.587      1.285      82.143       0.000
    E3              105.468      1.642      64.236       0.000
    E4              105.901      2.220      47.706       0.000
```

It can be seen that the Time 2, 3, and 4 means are each higher by about 5 IQ points compared to the Time-1 mean (which was estimated to be 100.107). The Wald test indicates at least one significant mean difference ($p < .001$):

```
MODEL FIT INFORMATION
. . .
Wald Test of Parameter Constraints
          Value                               99.494
          Degrees of Freedom                       3
          P-Value                             0.0000
```

Notice that the Wald test in our example has three degrees of freedom, as it simultaneously assesses all four means for equality (similar to an omnibus *F* statistic in analysis of variance). The means for Times 2, 3, and 4 do not appear to differ much from one another. Single degree of freedom Wald tests with appropriate corrections for alpha error inflation (e.g., Bonferroni) can be used to carry out post-hoc pairwise comparisons of means (e.g., of Time 1 vs. Time 2).

The fact that the unstandardized factor loadings increased over time in this example indicates that there was an increase in the true trait score variability across time. That is, true interindividual differences in intelligence increased. This can be seen by computing the variances of the trait variables at Times 2, 3, and 4, which are a function of the loadings and the initial (ξ_1) trait variance in this model:

$$Var(\xi_2) = \lambda_2^2 \cdot Var(\xi_1) = 1.517^2 \cdot 200.144 = 460.589$$
$$Var(\xi_3) = \lambda_3^2 \cdot Var(\xi_1) = 1.965^2 \cdot 200.144 = 772.801$$
$$Var(\xi_4) = \lambda_4^2 \cdot Var(\xi_1) = 2.679^2 \cdot 200.144 = 1{,}436.442$$

This increase in the variances corresponds to an increase in the corresponding standard deviations such that $SD(\xi_1) = 14.15$, $SD(\xi_2) = 21.46$, $SD(\xi_3) = 27.80$, and $SD(\xi_4) = 37.90$. This shows that some individuals' trait scores showed a larger increase than others', which would not have been permitted in the two previously discussed single-factor models. Nonetheless, even the ξ-congeneric model implies that individuals' trait scores remain perfectly correlated across time (the rank order of individuals does not change), which is a fairly restrictive assumption in many social science applications.

In the standardized solution (not shown here, but part of the materials available from the companion website [see the box at the end of the table of contents]), we can again inspect the standardized factor loadings, which tell us something about the reliabilities of the measurements as well as the extent to which our measurements really reflect a trait-like construct. In this example, all standardized loadings are again strong, indicating high reliabilities and showing that the construct is rather trait-like.

2.4.5 Summary

Among the three single-factor measurement models discussed, the ξ-congeneric model is the most general one. It estimates the largest number of free parameters and requires three time points to be just identified (saturated) and four or more time points to be overidentified with testable restrictions. The previous two models can be seen as special cases of the ξ-congeneric model: the random intercept model results when $\alpha_t = 0$ and $\lambda_t = 1$ for all $t = 1, \ldots, n$. The random and fixed intercepts model results when only $\lambda_t = 1$ for all $t = 1, \ldots, n$. All three models imply that constructs are trait-like. For such constructs, each model allows estimating the reliability of the measured variables at each time point.

In contrast to the two previously discussed single-factor measurement models, the ξ-congeneric model allows for interindividual differences in trait changes across time through the multiplicative parameter λ_t. However, the ξ-congeneric model remains rather restrictive with regard to interindividual differences in change across time. The model still requires the trait scores to remain perfectly correlated across time, implying a constant rank order of individuals across time.

2.5 CHAPTER SUMMARY

In this chapter, I discussed three simple longitudinal models and their application in Mplus. All models have in common that they (1) are based on just a single repeatedly measured variable and (2) use only a single latent factor.

Single-factor models offer a simple way of testing basic hypotheses about change processes. The single-factor random intercept model can be viewed as a baseline model for longitudinal studies, as it implies strict stability (no changes beyond what can be explained by random measurement error). The random and fixed intercepts model allows for changes in trait scores, but implies that those changes are strictly consistent across individuals (all people change by the same amount). The ξ-congeneric model allows for individual differences in the amount of trait changes but still assumes that individuals' trait scores maintain the same rank order across time.

The single-factor approaches have the advantage that they are mathematically identified already when there are fewer than four time points. Specifically, the random intercept model as well as the random and fixed intercepts models require only two measurement occasions ($n = 2$) to be identified. The random intercept model is overidentified with $df = 1$ for $n = 2$ time points, whereas the random and fixed intercepts model is just identified (saturated; $df = 0$) for $n = 2$. This means that the random intercept model already contains one testable restriction (the restriction of equal means across time) for a design with just two measurement occasions. The ξ-congeneric model requires three measurement occasions to be just identified ($df = 0$) and four or more measurement occasions to be overidentified with $df > 0$.

All three single-factor models have some important limitations. As mentioned above, the change processes implied by the models may be too restrictive for many empirical phenomena. Furthermore, the definitions of the models based on LST theory showed that each of the models implies a strict trait-like nature of the construct under study. In other words, no situational influences or person × situation interactions are allowed in the models. As a consequence, the models cannot be used to examine the degree to which measures may reflect situation and/or person × situation interaction influences, and they tend to underestimate the reliabilities of less than perfectly trait-like measures.

In the next chapter, I discuss models for single-indicator data that contain more than one latent variable. Models with multiple latent variables address some of the limitations of single-factor models. In particular, they allow for more flexibility in the modeling of individual differences in changes over time. Moreover, some multifactor models also allow for a separation of trait, state, and measurement error influences.

2.6 RECOMMENDED READING

Duncan, T., Duncan, S., & Strycker, L. (2006). *An introduction to latent variable growth curve modeling: Concepts, issues, and applications* (2nd ed.). Mahwah, NJ: Erlbaum.

NOTE

1. In addition to tests of statistical significance, researchers should also consider measures of effect size (e.g., Kline, 2020). This is because p-values in tests of significance are heavily influenced by sample size and contain no information about the practical significance of a given effect. In the hypothetical example, depending on the time interval, a raw mean difference of close to 5 IQ points over the course of a study may be seen as a rather large effect.